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Failure of the power laws on some subgraphs of the Z^2 lattice

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Abstract. We consider standard (Bernoulli) bond percolation on a subgraph G of Z^d. Denote by $\theta(p,G)$ and $\chi(p,G)$ the percolation probability and the mean cluster size, and $p_c(G) := \sup\{p : \theta(p,G) = 0\}$. It is widely believed that the power laws hold as follows:

$$\theta(p,G) - \theta(p_c(G),G) \approx (p - p_c(G))^{\beta} \qquad p > p_c(G)$$

and

$$\chi(p,G) \approx (p_c(G) - p)^{-\gamma} \qquad p < p_c(G)$$

for some positive constants β and γ which only depend on d. However, we find this conjecture is not true for some non-periodic subgraphs of Z^2.

1. Introduction and statement of results

We consider standard (Bernoulli) bond percolation on a subgraph G of Z^d in which all bonds are independently occupied with probability p and vacant with probability $1-p$. The corresponding probability measure on the configurations of occupied and vacant bonds is denoted by P_p. The cluster of the vertex x, $C(x,G)$, consists of all vertices which are connected to x by an occupied path on G. An occupied path is a nearest-neighbour path on G, all of whose bonds are occupied. By convention we always include x in $C(x,G)$. For any collection A of vertices, $|A|$ denotes the cardinality of A. The percolation probability is

$$\theta(p,G) = P_p(|C(0,G)| = \infty) \tag{1}$$

and the critical probability is

$$p_c(G) = \sup\{p : \theta(p,G) = 0\}.$$

Some other important functions are also defined as follows; the mean cluster size:

$$\chi(p,G) = E_p|C(0,G)| \tag{2}$$

the mean size of a finite cluster:

$$\chi^f(p,G) = E_p(|C(0,G)|; |C(0,G)| < \infty) \tag{3}$$

the correlation length:

$$\xi^{-1}(p,G) = \lim_{n\to\infty}\{-\tfrac{1}{n}\log P_p(0\leftrightarrow\partial B(n) \text{ on } G)\} \qquad \text{if } p < p_c(G) \tag{4}$$

or

$$\xi^{-1}(p,G) = \lim_{n\to\infty}\{-\tfrac{1}{n}\log P_p(0\leftrightarrow\partial B(n) \text{ on } G, |C(0,G)|<\infty)\} \quad \text{if } p \geqslant p_c(G) \tag{5}$$

and, finally, the number of clusters per vertex:

$$\kappa(p,G) = E_p(|C(0,G)|^{-1}) \tag{6}$$

where $B(n) = [-n,n]^d$, $\partial B(n) = \{x \in Z^d : \| x \| = n\}$, $E_p(X;A)$ is the mean of X on the event A, that is to say, $E_p(X;A) = E_p(XI_A)$, and $A \leftrightarrow B$ means that there exists an occupied path from some vertex of A to some vertex of B for any sets A and B.

It is clear that interesting phenomena occur when p is near to its critical value p_c. These concern largely the behaviour of θ, χ, χ^f, ξ and κ when $|p-p_c|$ is small. Indeed, it is widely believed (see e.g. Kesten 1982, 1987b, Grimmett 1989) that so called power laws and critical exponents exist on the Z^d lattice or on a general periodic subgraph G of Z^d (see the definition in Kesten H 1982). More precisely, the power laws are introduced as following:

$$\theta(p,G) - \theta(p_c(G),G) \approx (p-p_c(G))^{\beta} \text{ as } p \downarrow p_c(G) \tag{7}$$

$$\chi(p,G) \approx (p_c(G)-p)^{-\gamma} \text{ as } p \uparrow p_c(G) \tag{8}$$

$$\chi^f(p,G) \approx (p-p_c(G))^{-\gamma} \text{ as } p \downarrow p_c(G) \tag{9}$$

$$\xi(p,G) \approx |p_c(G)-p|^{-\nu} \text{ as } p \to p_c(G) \tag{10}$$

$$\kappa'''(p,G) \approx |p-p_c(G)|^{-1-\alpha} \text{ as } p \to p_c(G) \tag{11}$$

for some positive constants β, γ, ν and α which are called critical exponents. Here $f(p) \approx |p-p_c|^{\lambda}$ means that

$$C_1|p-p_c|^{\lambda} \leqslant f(p) \leqslant C_2|p-p_c|^{\lambda}$$

for some positive numbers C_1 and C_2. So far the power laws have been shown to hold partially only on high dimensional lattices (see Hara and Slade 1990). However, instead of the Z^d lattice or a periodic subgraph of Z^d, we shall concern ourselves with any subgraph of Z^d. Our central question then is whether the power laws still hold. Unfortunately, when considering percolation in some wedges which are non-periodic graphs, we find that the power laws no longer hold. Indeed, there are many interesting results in the studying of percolation on wedges (see e.g. Grimmett 1983, 1985, Chayes and Chayes 1986, Grimmett 1989). In particular, many strange phenomena are found on wedges such as that the percolation probability is discontinuous at the critical point. In the current paper, we shall show another strange phenomenon on wedges, that is, the failure of the power laws. More precisely, we shall think of G as a graph whose edges are bonds in

$$G(g) = \{x = (x_1,x_2) \in Z^2 : 0 \leqslant x_2 \leqslant g(x_1), x_1 \geqslant 0\} \tag{12}$$

where g is a specified function on $[0,\infty)$ taking non-negative values. Specifically, we are interested in the function $g(x) = x^a$ in (12) for some $a < 1$. Then it follows theorem 9.55 and theorem 9.12 in Grimmett (1989) that $p_c(G(x^a)) = \frac{1}{2}$ and $\theta(\frac{1}{2}, G(x^a)) = 0$. With the definition (12) we now state our main results.

Theorem 1. There exist constants $C_1(p) > 0$ and $\tau(p) > 0$ such that

$$\theta(p, G(x^a)) \leqslant C_1(p) \exp\left\{ - \left(\frac{1}{p - 1/2}\right)^{\tau(p)} \right\} \tag{13}$$

for all $p > \frac{1}{2}$.

Remark 1. (a) Equation (7) does not hold for $\theta(p, G(x^a))$.
(b) $\theta(p, G(x^a))$ is infinitely differentiable in p on $[0, 1]$.

Regarding the other parameters, we have:

Theorem 2. (a)$\chi^f(p, G(x^a))$ and $\kappa(p, G(x^a))$ are infinitely differentiable for all p and $\chi(p, G(x^a)) < \infty$ at $p = \frac{1}{2}$;
(b)$\xi(p, G(x^a)) = \xi(p, Z^2)$ if $p \leqslant \frac{1}{2}$ and $\xi(p, G(x^a)) = \infty$ if $p \geqslant \frac{1}{2}$.

Remark 2. Relations (8)–(11) are not true for $G(x^a)$.

2. Proofs

We introduce the definition of planar duality first. Define Z^* as the dual graph with vertex set $\{v + (\frac{1}{2}, \frac{1}{2})\}$ for $v \in Z^2$ and edges jointing all pairs of vertices which are one unit apart. For any bond set A, we write A^* for the corresponding dual bonds of A. For each bond $b^* \in Z^*$, we declare that b^* is occupied or vacant if b is occupied or vacant. In other words, each occupied (vacant) b^* crosses a corresponding occupied (vacant) bond in Z^2. By some geometric observations (see e.g. Kesten 1982), $B(n)$ is connected by an occupied path from its left side to its right side if and only if $B^*(n)$ cannot be connected by a vacant path from its top side to its bottom side. We call this 'the duality property'.

Proof of theorem 1. Define the occupied and the vacant crossing probabilities of $B(n)$ and $B^*(n)$ by

$$\sigma(p, n) = P_p(\exists \text{ an occupied path from the left edge to the right edge of } B(n))$$

$$\sigma^*(p, n) = P_p(\exists \text{ a vacant path from the top edge to the bottom edge of } B^*(n)).$$

The principal step we need is to show that the vacant crossing probabilities of squares are bounded away from zero when p is near $\frac{1}{2}$ from above. To make this precise, we first define

$$L(p) = \min\{n : \sigma(p, n) \geqslant 1 - \epsilon_0\} \qquad p > \tfrac{1}{2} \tag{14}$$

where ϵ_0 is some small, but strictly positive number whose precise value is not important. The important property is that ϵ_0 can be chosen such that there exists a constant δ for which

$$\sigma(p, n) \geqslant \delta \qquad \sigma^*(p, n) \geqslant \delta \tag{15}$$

uniformly in $n \leqslant L(p)$. It is shown in Kesten (1987a) that such a choice of ϵ_0 is possible. Note that the continuity of σ in p implies that

$$\sigma\left(\tfrac{1}{2}, n\right) \geqslant \delta \qquad \text{and} \qquad \sigma^*\left(\tfrac{1}{2}, n\right) \geqslant \delta \tag{16}$$

for all n. $L(p)$ is also called the correlation length. It is believed that

$$L(p) \approx \left(p - \tfrac{1}{2}\right)^{-\nu}.$$

However, we do not need such a sharp estimation. Indeed, equations (4.5) and (4.6) in Kesten (1987a) imply

$$L(p) \geqslant C_1 |p - \tfrac{1}{2}|^{-1/2} \tag{17}$$

for some constant $C_1 > 0$, which would suffice for our purposes here. With this knowledge, our theorem 1 becomes obvious. For each $p > \frac{1}{2}$, we choose n such that $n = C_1\left(p - \frac{1}{2}\right)^{-1/2}$. Clearly

$$\theta(p, G(x^a)) \leqslant P_p(D_n) \tag{18}$$

where D_n is the event that there is an occupied path from the origin to $\partial B(n)$ in $G(x^a)$. We follow the proof of theorem 9.55 in Grimmett (1989) to construct a sequence of boxes in the dual of Z^2. Define $\{w_k\}$ to be a sequence of vertices along the X-axis such that $w_0 = (x_0, y_0) = (0,0), w_1 = (x_1, y_1) = (1+2,0), w_2 = (x_2, y_2) = (x_1 + x_1^a + 2, 0), \cdots, w_i = (x_i, y_i) = (x_{i-1} + x_{i-1}^a + 2, 0), \cdots$ (see figure 1).

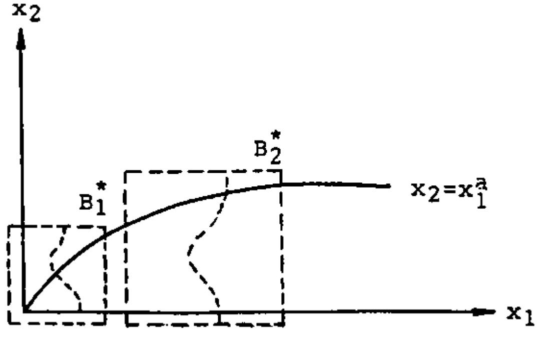

Figure 1. The existence of many vacant paths from the top to the bottom in the boxes $\{B_i^*\}$ prevents the formation of an occupied path in $G(x^a)$ from the origin to $\partial B(n)$.

Then we take

$$J = \max\{i : x_i \leqslant n\}.$$

Clearly

$$J \geqslant n^\beta \qquad \text{for some} \quad \beta > 0. \tag{19}$$

For each i, construct a square box B_i in Z^2 with side length x_i^a and with $w_{i-1}+1$ and $w_i - 1$ as the left and the right of the lower side. Therefore, the upper side of B_i lies strictly above the curve x^a (see figure 1). Let A_i be the event that B_i^* contains a vacant path from a vertex on its upper side to a vertex on its lower side. Thus, by the duality property, (15) and (17)

$$P_p(D_n) \leqslant (1 - P_p(A_i))^{n^\beta} \leqslant (1-\delta)^{C_1(p-1/2)^{-\beta/2}}. \tag{20}$$

In particular, by (16)

$$P_{\frac{1}{2}}(D_n) \leqslant (1-\delta)^{n^\beta} \tag{21}$$

for any n. Hence theorem 1 is proved by (20). □

Proof of remark 1. It follows from theorem 1 that remark 1(a) is self evident. Now we prove 1(b). Clearly, $\theta(p, G(x^a))$ is infinitely differentiable if $p \leqslant \frac{1}{2}$ by theorem 1. Specifically, it is a consequence of theorem 1 that

$$\theta^{(m)}(\tfrac{1}{2}, G(x^a)) = 0$$

for any m. Let F_n be the event that $C(0, G(x^a)) \cap \partial B(n) \neq \emptyset$ and $C(0, G(x^a)) \cap \partial B(n+1) = \emptyset$. Then

$$\theta(p, G(x^a)) = 1 - \sum_{k=0}^{\infty} P_p(F_k). \tag{22}$$

If F_n occurs, there exists a vacant cluster in the duality of Z^2 from some vertex of $\partial B(n+1)$ which contains at least n^a vertices. Therefore, it follows from (5.75) in Grimmett (1989) that

$$P_p(F_n) \leqslant n^a \exp(-Cn^a) \tag{23}$$

for some constant C if $p > \frac{1}{2}$. After that the infinite differentiablity of $\theta(p, G(x^a))$ can follow by the argument of Russo (1978) (see section 6.8 in Grimmett (1989) for details) if $p > \frac{1}{2}$. Therefore, remark 1(b) is proved. □

Proof of theorem 2. By (21) and (23), there exist constants $C > 0$ and $B > 0$ such that

$$P_p(F_n) \leqslant \exp(-Cn^B) \tag{24}$$

for all n and $p \geqslant \frac{1}{2}$. Since

$$\chi^f(p, G(x^a)) = \sum_k E_p(|C(0, G(x^a))|; |C(0, G(x^a))| < \infty \mid F_k) P_p(F_k)$$

and

$$E_p(|C(0, G(x^a))| \mid F_k) \leqslant n^{1+a}$$

we then can show the property of infinite differentiability of $\chi^f(p, G(x^a))$ by using (24) and the Russo's argument if $p \geqslant \frac{1}{2}$. When $p < \frac{1}{2}$, we can also follow the proof in section 5.4 of Grimmett (1989) to show the infinite differentiability of $\chi^f(p, G(x^a))$. Clearly, the same proof above also works for $\kappa(p, G(x^a))$. Furthermore, it follows from (21) again that $\chi(\frac{1}{2}, G(x^a)) < \infty$. Theorem 2(a) is thus proved.

Now we show that theorem 2(b) is true. First it can be seen that there exists a constant $C > 0$ such that

$$\begin{aligned} CP_p(0 \leftrightarrow \partial B(n) \text{ in } G(\log x)) \\ \leqslant P_p(0 \leftrightarrow \partial B(n) \text{ in } G(x^a)) \leqslant P_p(0 \leftrightarrow \partial B(n)). \end{aligned} \tag{25}$$

By theorem 5.2 in Chayes and Chayes 1986, we can deduce from (25) that $\xi(p, Z^2) = \xi(p, G(x^a))$ if $p < \frac{1}{2}$. On the other hand

$$\begin{aligned} p^{n^a+1}\theta(p, G(x^a)) \\ \leqslant P_p(\text{ all the bonds on } \{\partial B(n) \cap G(x^a)\}^* \text{ are vacant }) \\ \times P_p(0 \leftrightarrow \partial B(n) \text{ on } G(x^a)) \\ \leqslant P_p(0 \leftrightarrow \partial B(n) \text{ on } G(x^a), |C(0, G(x^a)| < \infty). \end{aligned} \tag{26}$$

Therefore, $\xi(p, G(x^a)) = \infty$ if $p > \frac{1}{2}$ since $a < 1$. When $p = \frac{1}{2}$, it follows the continuity of $\xi(p, Z^2)$ (see proposition 5.49 in Grimmett (1989)), (25) and (21) that

$$\xi\left(\tfrac{1}{2}, G(x^a)\right) = \xi\left(\tfrac{1}{2}, Z^2\right) = \infty.$$

Hence theorem 2(b) is also proved. □

Proof of remark 2. The failure of (8)–(11) is implied by theorem 2 directly. □

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